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On the chemical potential of D-instantons in $c = 0$ noncritical string theory

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Abstract: We study the chemical potential of D-instantons in $c = 0$ noncritical string theory. In a recent work [12], it was shown that the chemical potential can be calculated using the one matrix model. The calculation was done using the method of orthogonal polynomials and the authors obtained a universal value in the double scaling limit. We present an alternative method to calculate this value.

Keywords: D-branes, 2D Gravity, Matrix Models.

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1. Introduction

Noncritical string theory is a useful toy model in which we can study various aspects of critical string theory in a tractable way (for reviews, see, e.g. [1−4] and references therein). In particular, we can see that nonperturbative effects of string theory is of the form $\exp(-S_0/g_{\mathrm{s}})$ [5] from the analysis of noncritical string theory, where g_{s} is the string coupling constant. The value of S_0 can be deduced from the string equation or by using other techniques [6−8]. This effect can be considered to be due to D-instantons and S_0 is the classical action of the instanton [9−11]. As in the usual instanton analysis, one should calculate the chemical potential of the instanton to fix the magnitude of the nonperturbative effect. However, although the form of the nonperturbative effect can be obtained from the string equation, one cannot fix the normalization of it from this equation. For a long time, the value of the chemical potential of the instanton has been unknown.

Recently, in [12], the authors calculated the chemical potential for $c = 0$ noncritical string theory using the matrix model and showed that it is a universal quantity in the double scaling limit. This result was generalized to other noncritical string theories [13, 14] and was further discussed in [15]. In order to study the nonperturbative effects in the matrix model, it is convenient to consider the effective potential $V_{\mathrm{eff}}(z)$ for the eigenvalue z of the matrix. In [12], the authors calculated this potential using the method of orthogonal polynomials.

In this paper, we would like to present an alternative method to calculate the chemical potential. We only need information from the Schwinger-Dyson equations of the matrix model and obtain the same value for the chemical potential as that of [12]. Our method may be useful to calculate the chemical potential for more generic type of noncritical strings.

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The organization of this paper is as follows. In section 2, we discuss the effective potential $V_{\text{eff}}(z)$ of eigenvalues. We illustrate that if we try to compute $V_{\text{eff}}(z)$ using the usual genus expansion, the result in the next to leading order diverges when z is in the region where the eigenvalues are distributed. In section 3, we explain why the usual approximation breaks down for such z. In section 4, we present a method to calculate $V_{\text{eff}}(z)$ for z in the above mentioned region. The details of the calculations are given in the appendix. In section 5, using the results in section 4, we calculate the value of the chemical potential of the D-instanton and obtain the result which coincides with the one in [12]. Section 6 is devoted to the conclusions and discussions.

2. The effective potential of the matrix eigenvalues

$c = 0$ noncritical string theory can be analyzed by using the one matrix model

$$\int \mathrm{d}M \exp\left[-\frac{N}{g^2}\mathrm{Tr}\, V(M)\right], \tag{2.1}$$

where M is an $N \times N$ hermitian matrix and $V(M)$ is a polynomial of M. Taking the double scaling limit, one can calculate various quantities perturbatively with respect to the string coupling g_{s}. The matrix integral eq. (2.1) can be expressed by the integral over the eigenvalues z_i of M as

$$\int \prod_{i=1}^{N} \mathrm{d}z_i \exp\left[-\frac{N}{g^2}\sum_i V(z_i) + 2\,\mathrm{Re}\sum_{i>j}\ln(z_i - z_j)\right]. \tag{2.2}$$

In order to study the nonperturbative effects, it is convenient to define the effective potential $V_{\text{eff}}(z)$ of the matrix eigenvalues. If one picks up one eigenvalue, say z_N, and integrate over other eigenvalues putting $z_N = z$, one obtains

$$\int \prod_{i=1}^{N-1} \mathrm{d}z_i \exp\left[-\frac{N}{g^2}\sum_{i}^{N-1} V(z_i) + 2\,\mathrm{Re}\sum_{i>j}^{N-1}\ln(z_i - z_j) - \frac{N}{g^2}V(z) + 2\sum_i \ln|z - z_i|\right]. \tag{2.3}$$

This quantity can be regarded as the Boltzmann weight for z, and integrating it over z, we recover the above matrix integral. Therefore rewriting eq. (2.3) in terms of an $(N-1)\times(N-1)$ hermitian matrix M', we can define $V_{\text{eff}}(z)$ so that

$$\exp\left[-V_{\text{eff}}(z)\right] = \frac{\int \mathrm{d}M' \exp\left[-\frac{N-1}{g'^2}\mathrm{Tr}\, V(M') - \frac{N-1}{g'^2}V(z) + 2\,\mathrm{Re}\,\mathrm{Tr}\ln(z - M')\right]}{\int \mathrm{d}M' \exp\left[-\frac{N-1}{g'^2}\mathrm{Tr}\, V(M')\right]}, \tag{2.4}$$

where $g'^2 = g^2(1 - \frac{1}{N})$.

Naively, the right-hand side of the above formula can be calculated as

$$\exp\left\{-\frac{N-1}{g'^2}V(z) + 2\left\langle \mathrm{Re}\,\mathrm{Tr}\ln(z - M')\right\rangle + 2\left\langle \left[\mathrm{Re}\,\mathrm{Tr}\ln(z - M')\right]^2\right\rangle_c + \cdots\right\}, \tag{2.5}$$

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where

$$\langle F(M')\rangle = \frac{\int \mathrm{d}M' \exp\left[-\frac{N-1}{g'^2}\mathrm{Tr}\, V(M')\right] F(M')}{\int \mathrm{d}M' \exp\left[-\frac{N-1}{g'^2}\mathrm{Tr}\, V(M')\right]}, \tag{2.6}$$

and the subscript c denotes the connected part. Since insertions of $\mathrm{Tr}\ln(z - M')$ can be regarded as generating boundaries on the worldsheet, eq. (2.5) can be considered as a genus expansion of the free energy in open string theory. Therefore this expansion is natural, considering the relation between the D-branes and the eigenvalues in the matrix model [16, 17]. However, as we will see, this expansion is not always valid.

Eq. (2.5) seems to be valid in the leading order of $N \to \infty$ limit. Indeed, using this formula, the leading order contribution to $V_{\mathrm{eff}}(z)$ is given as

$$V_{\mathrm{eff}}(z) = -(N-1)\left[\frac{2}{N-1}\langle \mathrm{Re}\,\mathrm{Tr}\ln(z-M')\rangle - \frac{1}{g'^2}V(z)\right]. \tag{2.7}$$

An explicit form of this potential can be obtained once we know the resolvent $w(z) \equiv \frac{1}{N-1}\langle \mathrm{Tr}\frac{1}{z-M'}\rangle$ in the large-N limit. $w(z)$ can be obtained by solving the Schwinger-Dyson equation as

$$w(z) = \frac{1}{2g'^2}\left[V'(z) - M(z)\sqrt{(z-\alpha)(z-\beta)}\right], \tag{2.8}$$

where $M(z)$ is a polynomial. $M(z)$ and $\alpha,\ \beta$ ($\beta > \alpha$) are fixed by the condition $w(z) \sim 1/z$ when $z \to \infty$. Thus we get $V_{\mathrm{eff}}(z)$ as

$$\frac{N-1}{g'^2}\int^z \mathrm{d}z' M(z)\,\mathrm{Re}\sqrt{(z-\alpha)(z-\beta)}. \tag{2.9}$$

This potential is flat for the cut $\beta \geq z \geq \alpha$ of $w(z)$. This is consistent with the fact that the eigenvalues are distributed there. If we define $\rho(z)$ to be the distribution function of the eigenvalues of M', $w(z)$ can be expressed by $\rho(z)$ as

$$w(z) = \int \mathrm{d}z' \frac{\rho(z')}{z-z'}. \tag{2.10}$$

The inverse relation is $\rho(z) = -\frac{1}{\pi}\,\mathrm{Im}\,w(z)$. Therefore we can see that the eigenvalues are distributed in the cut $\beta \geq z \geq \alpha$ of $w(z)$.

$V_{\mathrm{eff}}(z)$ has a local maximum at $z = \gamma$ near $z = \beta$ and the configuration where one of the eigenvalues is at this point can be regarded as the D-instanton configuration.[1] D-instantons contribute to the partition function and one can evaluate such nonperturbative corrections to the free energy as [12]

$$\delta F = \frac{N\int_\beta^\infty \mathrm{d}z \exp\left[-V_{\mathrm{eff}}(z)\right]}{\int_\alpha^\beta \mathrm{d}z \exp\left[-V_{\mathrm{eff}}(z)\right]}. \tag{2.11}$$

N in the numerator is obtained by considering which of the N eigenvalues are at $z = \gamma$. In the double scaling limit, δF becomes of the form $\exp(-S_0/g_{\mathrm{s}} + \ln\mu_{\mathrm{inst.}} + \mathcal{O}(g_{\mathrm{s}}))$. S_0, $\mu_{\mathrm{inst.}}$

[1] In this paper, we assume that $V(z)$ is not an even polynomial, for simplicity.

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can be considered as the classical action and the chemical potential of the D-instanton respectively. One can evaluate S_0 from $V_{\rm eff}(z)$ in the large-N limit given above, and get the value consistent with the one from the string equation.

In order to get the chemical potential of the D-instanton, we should consider the next to leading order contribution in the large-N limit. If we trust eq. (2.5) what we need is the two loop correlator in the matrix model. It is known that [18]

$$\left\langle \mathrm{Tr}\frac{1}{z-M'}\mathrm{Tr}\frac{1}{z'-M'}\right\rangle_c = \frac{1}{4(z-z')^2}\left[\frac{2zz'-(\alpha+\beta)(z+z')+2\alpha\beta}{\sqrt{(z-\alpha)(z-\beta)}\sqrt{(z'-\alpha)(z'-\beta)}}-2\right]. \quad (2.12)$$

This quantity is universal in the sense that it depends on $V(z)$ only through the values of $\alpha,\ \beta$. Using this formula, we obtain

$$\langle \mathrm{Tr}\ln(z-M')\mathrm{Tr}\ln(z'-M')\rangle_c = -\ln\left(1-e^{-\theta-\theta'}\right), \quad (2.13)$$

where the variable θ is defined to be

$$e^{-\theta} = \frac{2}{\beta-\alpha}\left[z-\frac{\alpha+\beta}{2}-\sqrt{(z-\alpha)(z-\beta)}\right]. \quad (2.14)$$

Therefore the next to leading order contribution to $V_{\rm eff}(z)$ is given as

$$2\,\mathrm{Re}_\theta\,\mathrm{Re}_{\theta'}\ln(1-e^{-\theta-\theta'})\Big|_{z=z'}. \quad (2.15)$$

The behavior of this quantity depends crucially on whether z is inside the cut ($\beta\geq z\geq\alpha$) or outside the cut ($z>\beta$ or $\alpha>z$). If it is inside the cut, θ is imaginary and we get

$$\begin{aligned}&2\,\mathrm{Re}_\theta\,\mathrm{Re}_{\theta'}\ln(1-e^{-\theta-\theta'})\Big|_{z=z'}\\ &=\frac{1}{2}\ln\left[(1-e^{-\theta-\theta'})(1-e^{-\theta+\theta'})(1-e^{\theta-\theta'})(1-e^{\theta+\theta'})\right]\Big|_{\theta=\theta'}=\infty.\end{aligned} \quad (2.16)$$

Therefore the next to leading order contribution to $V_{\rm eff}(z)$ in eq. (2.5) diverges. In order to avoid this divergence, the authors of [12] used the method of orthogonal polynomials instead of the naive formula eq. (2.5). Actually such a divergence is necessary for eq. (2.11) to have a finite value in the double scaling limit. Since $N\to\infty$, we need an factor proportional to N in the denominator. Any way, we cannot use eq. (2.5) to calculate such a factor. For z outside the cut,

$$2\,\mathrm{Re}_\theta\,\mathrm{Re}_{\theta'}\ln(1-e^{-\theta-\theta'})\Big|_{z=z'} = 2\ln(1-e^{-2\theta}), \quad (2.17)$$

and we do not have such a problem.

The existence of the divergence indicates that the expansion eq. (2.5) is not a valid approximation for $\beta\geq z\geq\alpha$, while it has no problems for $z>\beta,\ \alpha>z$. We would like to discuss why the expansion eq. (2.5) is not valid for $\beta\geq z\geq\alpha$, in the next section.

3. Eq. (2.5) for $\beta \geq z \geq \alpha$

In order to understand the reason why eq. (2.5) is not valid for $\beta \geq z \geq \alpha$, let us study the matrix integral

$$\int \mathrm{d}M' \exp\left[-\frac{N-1}{g'^2}\mathrm{Tr}\, V(M') - \frac{N-1}{g'^2}V(z) + 2\,\mathrm{Re}\,\mathrm{Tr}\ln(z-M')\right], \tag{3.1}$$

without expanding the integrand as eq. (2.5). This can be considered as a matrix model with a matrix potential $\frac{N-1}{g'^2}\mathrm{Tr}V(M') - 2\,\mathrm{Re}\,\mathrm{Tr}\ln(z-M')$.

To study such a model, we consider the resolvent

$$\tilde{w}(z') \equiv \frac{1}{N-1}\left\langle \mathrm{Tr}\frac{1}{z'-M'}\right\rangle', \tag{3.2}$$

where $\langle\cdot\rangle'$ denotes the expectation value in this matrix model. Notice that $\tilde{w}(z')$ implicitly depends on z. The Schwinger-Dyson equation satisfied by the resolvent can be constructed in the usual way, and we obtain in the large-N limit,

$$\left(\tilde{w}(z')\right)^2 - \frac{1}{g'^2}V'(z')\tilde{w}(z') + \frac{1}{N-1}\left[\frac{\tilde{w}(z') - \tilde{w}(z+\mathrm{i}\epsilon)}{z'-z-\mathrm{i}\epsilon} + \frac{\tilde{w}(z') - \tilde{w}(z-\mathrm{i}\epsilon)}{z'-z+\mathrm{i}\epsilon}\right] = \frac{1}{4g'^4}f(z'), \tag{3.3}$$

where $f(z')$ is a polynomial of z'.

Now let us consider how the eigenvalues of M' are distributed in the matrix model eq. (3.1). Since there exists $-2\,\mathrm{Re}\,\mathrm{Tr}\ln(z-M')$ in the potential, eigenvalues cannot be distributed near z. Therefore $\tilde{w}(z)$ cannot have any imaginary part and $\tilde{w}(z+\mathrm{i}\epsilon) = \tilde{w}(z-\mathrm{i}\epsilon) = \tilde{w}(z)$. Thus, for z' not on the real axis, we get

$$(\tilde{w}(z'))^2 - \frac{1}{g'^2}V'(z')\tilde{w}(z') + \frac{2}{N-1}\frac{\tilde{w}(z')}{z'-z} = \frac{2}{N-1}\frac{\tilde{w}(z)}{z'-z} + \frac{1}{4g'^4}f(z'), \tag{3.4}$$

which can be solved as

$$\begin{aligned}\tilde{w}(z') &= \frac{1}{2g'^2}V'(z') - \frac{1}{N-1}\frac{1}{z'-z} + \\ &\quad + \sqrt{\frac{1}{4g'^4}\left[V'(z')^2 + f(z')\right] + \frac{2}{N-1}\frac{\tilde{w}(z)}{z'-z} + \left(\frac{1}{N-1}\frac{1}{z'-z}\right)^2}.\end{aligned} \tag{3.5}$$

In this expression we can see that for z' very close to z, $(\frac{1}{N-1}\frac{1}{z'-z})^2$ becomes very big and $\tilde{w}(z')$ has no imaginary part. In order to obtain $\tilde{w}(z')$ we should fix $f(z')$ and $\tilde{w}(z)$. Such quantities are usually determined by the conditions that $\tilde{w}(z)$ has singularities expected from the eigenvalue distributions.

The matrix model we are considering are made by picking up one eigenvalue of M and putting it at z. Therefore we expect that the distribution of the eigenvalues should be not so different from the original one, i.e. $\rho(z')$. When $z > \beta$, $\alpha > z$ this will be the case. Since z is outside the cut, the potential $-2\,\mathrm{Re}\,\mathrm{Tr}\ln(z-M')$ distorts the original distribution but the effect is of $\mathcal{O}(\frac{1}{N})$. Thus $f(z')$ and $\tilde{w}(z)$ can be fixed by the condition that $\tilde{w}(z')$ has one cut. On the other hand, for z with $\beta \geq z \geq \alpha$, the situation is quite different. Since there cannot be any eigenvalues near z, the original distribution is divided into two. Hence we should solve eq. (3.3) with the condition that $\tilde{w}(z')$ has two cuts.

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Actually we can expand the resolvent $\tilde{w}(z')$ as in eq. (2.5) as

$$\tilde{w}(z') = \sum_n \frac{1}{n!}\frac{1}{N-1}\left\langle \mathrm{Tr}\frac{1}{z'-M'}\left[2\,\mathrm{Re}\,\mathrm{Tr}\ln(z-M')\right]^n\right\rangle_c , \tag{3.6}$$

where $\langle\cdot\rangle$ is defined in eq. (2.6). If we expand $\tilde{w}(z')$ as

$$\tilde{w}(z') = \tilde{w}_0(z') + \frac{1}{N-1}\tilde{w}_1(z') + \frac{1}{(N-1)^2}\tilde{w}_2(z') + \cdots , \tag{3.7}$$

we can expand eq. (3.3) in terms of $\frac{1}{N-1}$ and we can see that $\tilde{w}_n(z')$ and $(N-1)^n \times \langle \mathrm{Tr}\frac{1}{z'-M'}(2\,\mathrm{Re}\,\mathrm{Tr}\ln(z-M'))^n\rangle_c$ satisfy the same equations. When $z > \beta$, $\alpha > z$, we can conclude that

$$\tilde{w}_n(z') = (N-1)^n\left\langle \mathrm{Tr}\frac{1}{z'-M'}\left[2\,\mathrm{Re}\,\mathrm{Tr}\ln(z-M')\right]^n\right\rangle_c , \tag{3.8}$$

because the condition that $\tilde{w}(z')$ has one cut coincides with the ones for $(N-1)^n \times \langle \mathrm{Tr}\,\frac{1}{z'-M'}[2\,\mathrm{Re}\,\mathrm{Tr}\ln(z-M')]^n\rangle_c$. Therefore we can treat the term $2\,\mathrm{Re}\,\mathrm{Tr}\ln(z-M')$ perturbatively. Since one can calculate the partition function from $\tilde{w}(z')$, we conclude that we can use eq. (2.5) in this case. However, for z with $\beta > z > \alpha$, this is not the case because the conditions for the cut is different. It implies that we cannot treat $2\,\mathrm{Re}\,\mathrm{Tr}\ln(z-M')$ perturbatively and therefore we cannot trust eq. (2.5) when $\beta \geq z \geq \alpha$.

To sum up, when $z > \beta$, $\alpha > z$ we can use eq. (2.5). For z with $\beta \geq z \geq \alpha$ we should not use eq. (2.5) which treats $2\,\mathrm{Re}\,\mathrm{Tr}\ln(z-M')$ perturbatively. In this case, for z' close to z the effect of the potential $2\,\mathrm{Re}\,\mathrm{Tr}\ln(z-M')$ is not perturbative. In principle, we can calculate $V_{\mathrm{eff}}(z)$ using the matrix model eq. (3.1) when $\beta \geq z \geq \alpha$.[2] However, since the next to leading order calculations of the matrix model eq. (3.1) are quite complicated, we resort to a different method.

4. $V_{\mathrm{eff}}(z)$ for $\beta \geq z \geq \alpha$

Since $V_{\mathrm{eff}}(z)$ cannot be calculated using eq. (2.5) for $\beta \geq z \geq \alpha$, we may not be able to trust the conclusions drawn from the leading order analysis in section 2. However, the z dependence of $V_{\mathrm{eff}}(z)$ can be found from the following arguments, and the properties of $V_{\mathrm{eff}}(z)$ given in section 2 turns out to be roughly true. Let us consider

$$\int \mathrm{d}M \exp\left[-\frac{N}{g^2}V(M)\right]\frac{1}{N}\mathrm{Tr}\frac{1}{z-M} . \tag{4.1}$$

Eq. (4.1) can be rewritten into an integral over matrix eigenvalues as

$$\int \prod_{i=1}^N \mathrm{d}z_i \exp\left[-\frac{N}{g^2}\sum V(z_i) + 2\,\mathrm{Re}\sum_{i>j}\ln(z_i - z_j)\right]\frac{1}{N}\sum_i \frac{1}{z-z_i} =$$

[2] We can fix the value of $\tilde{w}(z)$ from the condition that the number of the eigenvalues larger than z is the same as that of the original matrix model.

$$
\begin{aligned}
&= \int \prod_{i=1}^{N} \mathrm{d}z_i \exp\left[-\frac{N}{g^2}\sum V(z_i) + 2\,\mathrm{Re}\sum_{i>j}\ln(z_i - z_j)\right]\frac{1}{z - z_N} \\
&= C \int \mathrm{d}z' \exp\left[-V_{\text{eff}}(z')\right]\frac{1}{z - z'}\,, \qquad (4.2)
\end{aligned}
$$

where C is some constant. Since the distribution function of the eigenvalues of M can be obtained by taking the imaginary part of eq. (4.1) in the large-N limit, we can see that $\exp[-V_{\text{eff}}(z)]$ is proportional to this distribution function in the large-N limit. In the large-N limit, the distribution function for M coincides with that for M', i.e. $\rho(z)$. Indeed in [12] this fact was shown as a result of the calculation of $\exp[-V_{\text{eff}}(z)]$. Therefore we can conclude that $\exp[-V_{\text{eff}}(z)]$ is very small outside the cut. Since we can use eq. (2.5) outside the cut, $V_{\text{eff}}(z)$ possesses a local maximum near the cut and we can identify the D-instanton configuration.

In order to evaluate the chemical potential of the instanton, we need to calculate $\int_\alpha^\beta \mathrm{d}z \exp\left[-V_{\text{eff}}(z)\right] \sim \int \mathrm{d}z \exp\left[-V_{\text{eff}}(z)\right]$. Let us obtain this quantity through rather brute force calculations. From the definition eq. (2.4) we get

$$
\int \mathrm{d}z \exp\left[-V_{\text{eff}}(z)\right] = \frac{\int \mathrm{d}z \int \mathrm{d}M' \exp\left[-\frac{N-1}{g'^2}\mathrm{Tr}\,V(M') - \frac{N-1}{g'^2}V(z) + 2\,\mathrm{Re}\,\mathrm{Tr}\ln(z - M')\right]}{\int \mathrm{d}M' \exp\left[-\frac{N-1}{g'^2}\mathrm{Tr}V(M')\right]}\,. \qquad (4.3)
$$

Since the numerator is proportional to the partition function $\int \mathrm{d}M \exp\left[-\frac{N}{g^2}\mathrm{Tr}V(M)\right]$, what we should calculate is essentially the ratio between the matrix integrals over M and M'. These matrix integrals can be evaluated using $\rho(z)$ and $w(z)$ in the large-N limit. After some calculations, which are presented in the appendix, we obtain

$$
\int \mathrm{d}z \exp\left[-V_{\text{eff}}(z)\right] \cong \frac{\pi(N-1)}{2}(\beta - \alpha)\exp\left[2(N-1)R\right]\,, \qquad (4.4)
$$

where

$$
R \equiv \lim_{\Lambda\to\infty}\left[\int_\Lambda^\beta dz' w(z') + \ln\Lambda\right] - \frac{1}{2g'^2}V(\beta)\,. \qquad (4.5)
$$

From eq. (4.4), we can see that for $\beta \geq z \geq \alpha$,

$$
\exp\left[-V_{\text{eff}}(z)\right] \cong \frac{\pi(N-1)}{2}(\beta - \alpha)\exp\left[2(N-1)R\right]\rho(z)\,. \qquad (4.6)
$$

5. The chemical potential of the instanton

Now that we know $V_{\text{eff}}(z)$ when $\beta \geq z \geq \alpha$, we can calculate the chemical potential of the instanton. For z larger than β we can use the formula eq. (2.5) and we obtain

$$
\exp\left[-V_{\text{eff}}(z)\right] \cong \frac{1}{(1 - e^{-2\theta})^2}\exp\left[2(N-1)R - \frac{N-1}{g'^2}\int_\beta^z dz' M(z')\sqrt{(z'-\alpha)(z'-\beta)}\right]\,, \qquad (5.1)
$$

in the large-N limit.

The integral $\int_\beta^\infty \mathrm{d}z \exp\left[-V_{\rm eff}(z)\right]$ can be evaluated using the saddle point approximation. The saddle point we are interested in is $z=\gamma$ near the cut, at which $V_{\rm eff}(z)$ has a local maximum. In the large-N limit,

$$V_{\rm eff}(z) \sim \frac{N-1}{g'^2}\int_\beta^z \mathrm{d}z' M(z')\sqrt{(z'-\alpha)(z'-\beta)}\,, \tag{5.2}$$

and the polynomial $M(z')$ is factorized as

$$-\frac{1}{g'^2}M(z') = G(z')(z'-\gamma)\,. \tag{5.3}$$

Then

$$\begin{aligned}\int_\beta^\infty \mathrm{d}z \exp\left[-V_{\rm eff}(z)\right] \cong\ & \mathrm{i}\sqrt{\frac{2\pi}{(N-1)G(\gamma)\left[(\gamma-\alpha)(\gamma-\beta)\right]^{\frac{5}{2}}}}\,\cdot \\ & \cdot\frac{(\beta-\alpha)^4}{64\left[\gamma-(\alpha+\beta)/2-\sqrt{(\gamma-\alpha)(\gamma-\beta)}\right]^2}\times \\ & \times\exp\left[2(N-1)R-\frac{N-1}{g'^2}\int_\beta^\gamma \mathrm{d}z' M(z')\sqrt{(z'-\alpha)(z'-\beta)}\right].\end{aligned} \tag{5.4}$$

Thus the instanton contribution to the partition function is given as

$$\begin{aligned}\delta F &= \frac{N\int_\beta^\infty \mathrm{d}z\exp\left[-V_{\rm eff}(z)\right]}{\int \mathrm{d}z\exp\left[-V_{\rm eff}(z)\right]} \\ &\cong \frac{\mathrm{i}(\beta-\alpha)^3}{16\sqrt{2\pi}\left[\gamma-(\alpha+\beta)/2-\sqrt{(\gamma-\alpha)(\gamma-\beta)}\right]^2\sqrt{NG(\gamma)\left[(\gamma-\alpha)(\gamma-\beta)\right]^{\frac{5}{2}}}}\times \\ &\quad\times\exp\left[-\frac{N-1}{g'^2}\int_\beta^\gamma dz' M(z')\sqrt{(z'-\alpha)(z'-\beta)}\right].\end{aligned} \tag{5.5}$$

In the continuum limit, we take the limit in which g'^2 goes to the critical value $g_c'^2$, which satisfies $\beta(g_c'^2)=\gamma(g_c'^2)=z_c$. More explicitly, we take the limit $a\to 0$ so that

$$g_c'^2 - g'^2 \propto \mu a^2\,, \tag{5.6}$$

where μ is the cosmological constant. Then with an appropriate normalization of μ,

$$\begin{aligned}\beta &\cong z_c - \sqrt{\mu}a\,,\\ \gamma &\cong z_c + \frac{1}{2}\sqrt{\mu}a\,.\end{aligned} \tag{5.7}$$

In this limit, the classical action part of the instanton becomes

$$\frac{N-1}{g'^2}\int_\beta^\gamma dz' M(z')\sqrt{(z'-\alpha)(z'-\beta)} \cong \frac{8\sqrt{3}}{5}\mu^{\frac{5}{4}}\left(3\cdot 2^{-\frac{7}{2}}NG(z_c)\sqrt{z_c-\alpha}a^{\frac{5}{2}}\right). \tag{5.8}$$

Therefore if we take the double scaling limit $N\to\infty$, $a\to 0$ with

$$3\cdot 2^{-\frac{7}{2}}NG(z_c)\sqrt{z_c-\alpha}\,a^{\frac{5}{2}} = \frac{1}{g_{\rm s}}\,, \tag{5.9}$$

fixed, we get $8\sqrt{3}\mu^{\frac{5}{4}}/5g_{\rm s}$ for the classical action. Then the instanton contribution to the partition function becomes

$$\begin{aligned}\delta F &\cong \frac{\mathrm{i}}{4\sqrt{2\pi}\left[NG(z_c)\sqrt{z_c-\alpha}\left(\frac{3}{2}\sqrt{\mu}a\right)^{\frac{5}{2}}\right]^{\frac{1}{2}}}\exp\left(-\frac{8\sqrt{3}\mu^{\frac{5}{4}}}{5g_{\rm s}}\right)\\ &= \frac{ig_{\rm s}^{\frac{1}{2}}}{8\cdot 3^{\frac{3}{4}}\cdot\sqrt{\pi}\mu^{\frac{5}{8}}}\exp\left(-\frac{8\sqrt{3}\mu^{\frac{5}{4}}}{5g_{\rm s}}\right). \end{aligned} \tag{5.10}$$

Thus we have the chemical potential as a universal quantity which coincides with the one given in [12].

6. Conclusions and discussions

In this paper we calculate the value of the chemical potential of the D-instanton and obtain the universal value obtained in [12]. The problem was that the next to leading order contribution in eq. (2.5) is divergent for $\beta \geq z \geq \alpha$. We discuss that this is because for z in such a region the existence of the eigenvalue affects the distributions of other eigenvalues which cannot be treated perturbatively.

We should mention that although eq. (2.5) is valid when $z > \beta$, $\alpha > z$, we should take some care for z close to α, β. If one wants to calculate $V_{\rm eff}(z)$ for some fixed z in the large-N limit, eq. (2.5) can be used. However, if one wants to know the value of $V_{\rm eff}(z)$ for $z = \beta + \mathcal{O}(\frac{1}{N})$ for example, this equation may not be used. As can be seen from eq. (2.17) the next to leading order contribution to $V_{\rm eff}(z)$ diverges in this case. Any way, we do not have to be annoyed by these matters because we are interested in $V_{\rm eff}(z)$ around $z = \gamma$.

Since we need only information from the Schwinger-Dyson equations to calculate the chemical potential in our method, we will be able to calculate it in other noncritical string theories. We would like to study this problem elsewhere.

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A. Eq. (4.4)

In this appendix, we will explain how to calculate $\int \mathrm{d}z \exp\left[-V_{\rm eff}(z)\right]$ and get the result eq. (4.4) in the large-N limit.

In order to calculate $\int \mathrm{d}z \exp\left[-V_{\rm eff}(z)\right]$ to the next to leading order, we should be a little bit accurate. Let us fix the normalization of the integration measure $\mathrm{d}M$ and $\mathrm{d}M'$ so that

$$\begin{aligned}\int \mathrm{d}M \exp\left[-\frac{N}{2g^2}\mathrm{Tr}\, M^2\right] &= 1\,,\\ \int \mathrm{d}M' \exp\left[-\frac{N-1}{2g'^2}\mathrm{Tr} M'^2\right] &= 1\,. \end{aligned} \tag{A.1}$$

Then the matrix integrations are related to the integrations over the matrix eigenvalues as

$$\begin{aligned}\int \mathrm{d}M &= C_N^{-1}\int \prod_{i=1}^{N} \mathrm{d}z_i \prod_{i>j}(z_i - z_j)^2,\\ \int \mathrm{d}M' &= C_{N-1}^{-1}\int \prod_{i=1}^{N-1} \mathrm{d}z'_i \prod_{i>j}(z'_i - z'_j)^2,\end{aligned} \tag{A.2}$$

where

$$\begin{aligned}C_N &= N!(2\pi)^{\frac{N}{2}}\left(\frac{g^2}{N}\right)^{\frac{N^2}{2}} \prod_{k=1}^{N-1} k^{N-k},\\ C_{N-1} &= (N-1)!(2\pi)^{\frac{N-1}{2}}\left(\frac{g'^2}{N-1}\right)^{\frac{(N-1)^2}{2}} \prod_{k=1}^{N-2} k^{N-1-k}.\end{aligned} \tag{A.3}$$

Now eq. (4.3) becomes

$$\int \mathrm{d}z \exp\left[-V_{\mathrm{eff}}(z)\right] = \frac{C_N \int \mathrm{d}M \exp\left[-\frac{N}{g^2}\mathrm{Tr}\, V(M)\right]}{C_{N-1}\int \mathrm{d}M' \exp\left[-\frac{N-1}{g'^2}\mathrm{Tr}\, V(M')\right]}. \tag{A.4}$$

With the normalization conditions eq. (A.1), the partition functions can be expanded in the large-N limit as

$$\begin{aligned}\int \mathrm{d}M \exp\left[-\frac{N}{g^2}\mathrm{Tr}\, V(M)\right] &= \exp\left[N^2 F_0(g^2) + F_1(g^2) + \cdots\right],\\ \int \mathrm{d}M' \exp\left[-\frac{N-1}{g'^2}\mathrm{Tr}\, V(M')\right] &= \exp\left[(N-1)^2 F_0(g'^2) + F_1(g'^2) + \cdots\right],\end{aligned} \tag{A.5}$$

where F_n are contributions from the Feynman diagrams with the topology of genus n Riemann surface. Therefore in the large-N limit,

$$\begin{aligned}\int \mathrm{d}z \exp\left[-V_{\mathrm{eff}}(z)\right] = \frac{C_N}{C_{N-1}}\exp\Bigg\{&(N-1)\left[2F_0(g'^2) + g'^2 F_0'(g'^2)\right] + F_0(g'^2) +\\ &+ 2g'^2 F_0'(g'^2) + \frac{1}{2}g'^4 F_0''(g'^2) + \mathcal{O}\left(\frac{1}{N}\right)\Bigg\}.\end{aligned} \tag{A.6}$$

$F_0(g'^2)$ can be evaluated using $\rho(z)$ in eq. (2.10). Since

$$\begin{aligned}\int \mathrm{d}M' \exp\left[-\frac{N-1}{g'^2}\mathrm{Tr}\, V(M')\right] &= C_{N-1}^{-1}\int \prod_{i=1}^{N-1}\mathrm{d}z'_i \prod_{i>j}(z'_i - z'_j)^2 \exp\left[-\frac{N-1}{g'^2}\sum V(z'_i)\right]\\ &= \exp\Bigg\{(N-1)^2\Bigg[\int \mathrm{d}z \int \mathrm{d}z' \rho(z)\rho(z') \ln|z - z'| -\\ &\qquad - \frac{1}{g'^2}\int \mathrm{d}z \rho(z) V(z) + \frac{3}{4} - \frac{1}{2}\ln g'^2\Bigg] +\\ &\qquad + \mathcal{O}(N^0)\Bigg\},\end{aligned} \tag{A.7}$$

we get

$$F_0(g'^2) = \int \mathrm{d}z \int \mathrm{d}z' \rho(z)\rho(z') \ln|z-z'| - \frac{1}{g'^2} \int \mathrm{d}z \rho(z) V(z) + \frac{3}{4} - \frac{1}{2} \ln g'^2 \,. \tag{A.8}$$

From eq. (2.10), we obtain

$$\int \mathrm{d}z' \rho(z') \ln|z-z'| = \lim_{\Lambda\to\infty} \left[\int_\Lambda^z \mathrm{d}z' \,\mathrm{Re}\, w(z') + \ln\Lambda \right]. \tag{A.9}$$

Substituting this formula into eq. (A.8), taking the form of $w(z)$ in eq. (2.8) into account, we get

$$F_0(g'^2) = -\frac{1}{2g'^2} \int \mathrm{d}z \rho(z) V(z) + R + \frac{3}{4} - \frac{1}{2} \ln g'^2 \,, \tag{A.10}$$

where R is defined in eq. (4.5).

In order to obtain $\int \mathrm{d}z \exp\left[-V_{\mathrm{eff}}(z)\right]$, we should calculate $2F_0(g'^2) + g'^2 F_0'(g'^2)$ and $F_0(g'^2) + 2g'^2 F_0'(g'^2) + \frac{1}{2} g'^4 F_0''(g'^2)$. After substituting eq. (A.10) into these, it is convenient to proceed as follows. From the relation

$$\begin{aligned} g'^2 \partial_{g'^2} \ln & \left\{ \int \prod_{i=1}^{N-1} \mathrm{d}z_i' \prod_{i>j} (z_i' - z_j')^2 \exp\left[-\frac{N-1}{g'^2} \sum V(z_i') \right] \right\} = \\ & = \frac{(N-1)^2}{g'^2} \int \mathrm{d}z \rho(z) V(z) + \mathcal{O}((N-1)^0) \,, \end{aligned} \tag{A.11}$$

one can prove the following identity:

$$g'^2 \partial_{g'^2} \left[R - \frac{1}{2g'^2} \int \mathrm{d}z \rho(z) V(z) \right] = \frac{1}{g'^2} \int \mathrm{d}z \rho(z) V(z) \,. \tag{A.12}$$

Using this identity, we can express the two quantities above in terms of R as

$$2F_0(g'^2) + g'^2 F_0'(g'^2) = 2R + 1 - \ln g'^2 \,, \tag{A.13}$$

$$F_0(g'^2) + 2g'^2 F_0'(g'^2) + \frac{1}{2} g'^4 F_0''(g'^2) = R + g'^2 \partial_{g'^2} R - \frac{1}{2} \ln g'^2 \,. \tag{A.14}$$

From eq. (A.13), one can get the leading order part of eq. (4.4).

To further simplify the right hand side of eq. (A.14), we use the following expression for R:

$$R = \frac{1}{N-1} \left\langle \mathrm{Tr} \ln(\beta - M') \right\rangle - \frac{1}{2g'^2} V(\beta) \,. \tag{A.15}$$

From this expression, we get

$$\begin{aligned} g'^2 \partial_{g'^2} R &= \frac{1}{2g'^2} V(\beta) + \left\langle \frac{1}{g'^2} \mathrm{Tr}\, V(M') \mathrm{Tr} \ln(\beta - M') \right\rangle_c + \\ & \quad + \left[\frac{1}{N-1} \left\langle \mathrm{Tr} \frac{1}{\beta - M'} \right\rangle - \frac{1}{2g'^2} V'(\beta) \right] g'^2 \partial_{g'^2} \beta \\ &= \frac{1}{2g'^2} V(\beta) + \left\langle \frac{1}{g'^2} \mathrm{Tr}\, V(M') \mathrm{Tr} \ln(\beta - M') \right\rangle_c \,, \end{aligned} \tag{A.16}$$

where we used eq. (2.8) in the last line. Therefore we obtain the following expression for $R + g'^2 \partial_{g'^2} R$:

$$
\begin{aligned}
R + g'^2 \partial_{g'^2} R &= \frac{1}{N-1} \langle \mathrm{Tr} \ln(\beta - M') \rangle + \left\langle \frac{1}{g'^2} \mathrm{Tr}\, V(M') \mathrm{Tr} \ln(\beta - M') \right\rangle_c \qquad \text{(A.17)} \\
&= \lim_{\Lambda \to \infty} \left\{ \int_\Lambda^\beta \mathrm{d}z \left[w(z) + \oint_C \frac{\mathrm{d}z'}{2\pi i} \frac{1}{g'^2} V(z') \left\langle \mathrm{Tr} \frac{1}{z - M'} \mathrm{Tr} \frac{1}{z' - M'} \right\rangle_c \right] + \ln \Lambda \right\}.
\end{aligned}
$$

Here we have chosen C to be a closed contour in the z plane, which encloses the cut $[\alpha, \beta]$ leaving outside the point z'.

Now let us substitute the correlation functions $w(z)$ and $\langle \mathrm{Tr} \frac{1}{z-M'} \mathrm{Tr} \frac{1}{z'-M'} \rangle$ of the matrix model into this formula. It is known that $w(z)$ has the following expression [19, 20]:

$$
w(z) = \oint_C \frac{\mathrm{d}z'}{4\pi i} \frac{1}{g'^2} \frac{V'(z')}{z - z'} \sqrt{\frac{(z-\alpha)(z-\beta)}{(z'-\alpha)(z'-\beta)}}\,. \qquad \text{(A.18)}
$$

On the other hand, $\left\langle \mathrm{Tr} \frac{1}{z-M'} \mathrm{Tr} \frac{1}{z'-M'} \right\rangle_c$ in eq. (2.12) can be rewritten as

$$
\left\langle \mathrm{Tr} \frac{1}{z - M'} \mathrm{Tr} \frac{1}{z' - M'} \right\rangle_c = -\frac{1}{2} \partial_{z'} \left[\frac{1}{z - z'} \left(1 - \sqrt{\frac{(z'-\alpha)(z'-\beta)}{(z-\alpha)(z-\beta)}} \right) \right]. \qquad \text{(A.19)}
$$

Moreover, it is known that the following identities hold [19, 20]:

$$
\oint_C \frac{\mathrm{d}z'}{2\pi i} \frac{1}{g'^2} \frac{V'(z')}{\sqrt{(z'-\alpha)(z'-\beta)}} = 0\,, \qquad \text{(A.20)}
$$

$$
\oint_C \frac{\mathrm{d}z'}{2\pi i} \frac{1}{g'^2} \frac{z' V'(z')}{\sqrt{(z'-\alpha)(z'-\beta)}} = 2\,. \qquad \text{(A.21)}
$$

Substituting the expressions for the correlators above into eq. (A.17), taking the identities eqs. (A.20)(A.21) into account, we eventually get

$$
\begin{aligned}
R + g'^2 \partial_{g'^2} R &= \lim_{\Lambda \to \infty} \left[\int_\Lambda^\beta \mathrm{d}z \frac{1}{\sqrt{(z-\alpha)(z-\beta)}} + \ln \Lambda \right] \\
&= \ln \frac{\beta - \alpha}{4}\,. \qquad \text{(A.22)}
\end{aligned}
$$

Substituting all these results into eq. (A.6), we get

$$
\int \mathrm{d}z \exp\left[-V_{\mathrm{eff}}(z)\right] \cong \frac{\pi(N-1)}{2} (\beta - \alpha) \exp\left[2(N-1)R\right]. \qquad \text{(A.23)}
$$

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